**Physics 531 – Problem Set 2** **R. McDermott**
**Due in class Monday Feb. 11**

1. Griffiths Problem 2.1.
2. Griffiths Problem 2.2.
3. Griffiths Problem 2.5.
4. Griffiths Problem 2.6.
5. Griffiths Problem 2.7.
6. Griffiths Problem 2.8.
7. Griffiths Problem 2.9.

**Problem 2.1** Prove the following three theorems:

**(a)** For normalizable solutions, the separation constant $E$ must be *real*. *Hint:* Write $E$ (in Equation 2.7) as $E_0 + i\Gamma$ (with $E_0$ and $\Gamma$ real), and show that if Equation 1.20 is to hold for all $t$, $\Gamma$ must be zero.

**(b)** The time-independent wave function $\psi(x)$ can always be taken to be *real* (unlike $\Psi(x, t)$, which is necessarily complex). This doesn't mean that every solution to the time-independent Schrödinger equation *is* real; what it says is that if you've got one that is *not*, it can always be expressed as a linear combination of solutions (with the same energy) that *are*. So you *might as well* stick to $\psi$s that are real. *Hint:* If $\psi(x)$ satisfies Equation 2.5, for a given $E$, so too does its complex conjugate, and hence also the real linear combinations $(\psi + \psi^*)$ and $i\,(\psi - \psi^*)$.

**(c)** If $V(x)$ is an **even function** (that is, $V(-x) = V(x)$) then $\psi(x)$ can always be taken to be either even or odd. *Hint:* If $\psi(x)$ satisfies Equation 2.5, for a given $E$, so too does $\psi(-x)$, and hence also the even and odd linear combinations $\psi(x) \pm \psi(-x)$.

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#3 + #4 30
#5 + #6 30
#7 15

$\boxed{100/100}$

Homework 2

2.1) a) $\Psi(x,t) = \psi(x) e^{-i(E_0 + i\Gamma)t/\hbar}$ $= \psi(x) e^{-iE_0 t/\hbar} e^{\Gamma t/\hbar}$

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = \int_{-\infty}^{\infty} \Psi \Psi^* dx$$

$$= \int_{-\infty}^{\infty} \psi(x) e^{2\Gamma t/\hbar} dx$$

$\psi(x)$ doesn't depend on $t$, so for $\int_{-\infty}^{\infty} |\Psi|^2 dx = 1$ for all $t$, $\boxed{\Gamma = 0}$

b) If $\underset{\text{complex}}{\tilde{\psi}(x)}$ satisfies $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$ so does $\psi^*$, $\underbrace{(\psi+\psi^*)}_{2Re[\psi]}$ and $\underbrace{i(\psi-\psi^*)}_{2Im[\psi]}$.

$$\tilde{\psi}(x) = Re[\psi] + Im[\psi] i$$
$$= \frac{1}{2}(\psi + \psi^*) + i^2 \frac{1}{2}(\psi - \psi^*)$$

Since $\tilde{\psi}$ is a linear combination of real solutions, we can always treat $\psi$ as real.

c) $V(x) = V(-x)$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = E\psi$$

$\psi(-x)$ also is a solution.

$\psi(x) + \psi(-x)$ and $\psi(x) - \psi(-x)$ are also solutions.

even odd

$$\psi = \frac{1}{2}(\psi(x) + \psi(-x)) + \frac{1}{2}(\psi(x) - \psi(-x))$$

Since $\psi$ is a linear combination of $\psi(x) + \psi(-x)$ and $\psi(x) - \psi(-x)$, $\psi$ is either even or odd.

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**Problem 2.2** Show that $E$ must exceed the minimum value of $V(x)$, for every normalizable solution to the time-independent Schrödinger equation. What is the classical analog to this statement? *Hint:* Rewrite Equation 2.5 in the form

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi;$$

if $E < V_{\min}$, then $\psi$ and its second derivative always have the *same sign*—argue that such a function cannot be normalized.

**Problem 2.5** A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:

$$\Psi(x, 0) = A\left[\psi_1(x) + \psi_2(x)\right].$$

**(a)** Normalize $\Psi(x, 0)$. (That is, find $A$. This is very easy, if you exploit the orthonormality of $\psi_1$ and $\psi_2$. Recall that, having normalized $\Psi$ at $t = 0$, you can rest assured that it *stays* normalized—if you doubt this, check it explicitly after doing part (b).)

**(b)** Find $\Psi(x, t)$ and $|\Psi(x, t)|^2$. Express the latter as a sinusoidal function of time, as in Example 2.1. To simplify the result, let $\omega \equiv \pi^2\hbar/2ma^2$.

**(c)** Compute $\langle x \rangle$. Notice that it oscillates in time. What is the angular frequency of the oscillation? What is the amplitude of the oscillation? (If your amplitude is greater than $a/2$, go directly to jail.)

**(d)** Compute $\langle p \rangle$. (As Peter Lorre would say, "Do it ze *kveek* vay, Johnny!")

**(e)** If you measured the energy of this particle, what values might you get, and what is the probability of getting each of them? Find the expectation value of $H$. How does it compare with $E_1$ and $E_2$?

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**Problem 2.6** Although the *overall* phase constant of the wave function is of no physical significance (it cancels out whenever you calculate a measurable quantity), the *relative* phase of the coefficients in Equation 2.17 *does* matter. For example, suppose we change the relative phase of $\psi_1$ and $\psi_2$ in Problem 2.5:

$$\Psi(x, 0) = A\left[\psi_1(x) + e^{i\phi}\psi_2(x)\right],$$

where $\phi$ is some constant. Find $\Psi(x, t)$, $|\Psi(x, t)|^2$, and $\langle x \rangle$, and compare your results with what you got before. Study the special cases $\phi = \pi/2$ and $\phi = \pi$. (For a graphical exploration of this problem see the applet in footnote 9 of this chapter.)

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**Problem 2.7** A particle in the infinite square well has the initial wave function

$$\Psi(x, 0) = \begin{cases} Ax, & 0 \le x \le a/2, \\ A(a - x), & a/2 \le x \le a. \end{cases}$$

**(a)** Sketch $\Psi(x, 0)$, and determine the constant $A$.

**(b)** Find $\Psi(x, t)$.

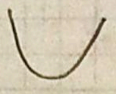
**(c)** What is the probability that a measurement of the energy would yield the value $E_1$?

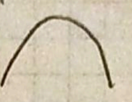
**(d)** Find the expectation value of the energy, using Equation 2.21.[21]

2.2) $\frac{\partial^2\psi}{\partial x^2} = \frac{2m}{\hbar^2}(V(x)-E)\psi$

If $E < V_{min}$, $\frac{\partial^2\psi}{\partial x^2}$ is always either positive or negative.

i.e. curving up or curving down





if we integrate from $-\infty$ to $\infty$, the answer will always diverge because $\psi \to \pm\infty$ as $x \to \pm\infty$.

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2.5) a) $1 = \int_{-\infty}^{\infty} A^2 |\psi_1(x) + \psi_2(x)|^2 dx = A^2 \int_{-\infty}^{\infty} [\psi_1(x)^2 + \psi_2(x)^2 + \psi_1^*(x)\psi_2(x) + \psi_1(x)\psi_2^*(x)] dx$

$= A^2(1+1)$

$$\boxed{A = \frac{1}{\sqrt{2}}}$$

b) $c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) A[\psi_1(x) + \psi_2(x)]$

$= \frac{1}{\sqrt{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right) dx$

$= \frac{1}{\sqrt{a}}\left(\frac{a}{2}\delta_{n1} + \frac{a}{2}\delta_{n2}\right)$

$= \begin{cases} \frac{\sqrt{a}}{2} & \text{if } n = 1, 2 \\ 0 & \text{if } n > 2 \end{cases}$

$\omega = \frac{\pi^2\hbar}{2ma^2}$

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{a}} \sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \frac{1}{\sqrt{a}} \sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}} \quad (2.34)$$

$|\Psi(x,t)|^2 = \frac{1}{a}\left(\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}\right)\left(\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{4i\omega t}\right)$

$= \frac{1}{a}\left(\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{3i\omega t} + e^{-3i\omega t}\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right)$

$$\boxed{= \frac{1}{2a}\left(\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2\cos(3\omega t)\right)}$$

c) $$\langle x\rangle = \int_0^a \frac{x}{a}\left(\sin^2\left(\frac{\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t) + \sin^2\left(\frac{2\pi}{a}x\right)\right)dx$$

$u = x \quad v = \frac{x}{2} - \frac{a\sin\frac{2\pi}{a}x}{4\pi}$ $\qquad u = x \quad v = \frac{x}{2} - \frac{a\sin\frac{4\pi}{a}x}{8\pi}$

$du = dx \quad dv = \sin^2\left(\frac{\pi}{a}x\right)$ $\qquad du = dx \quad dv = \sin^2\left(\frac{2\pi}{a}x\right)$

$$= \frac{1}{a}\left[\left[\frac{x^2}{2} - \frac{xa\sin\frac{2\pi}{a}x}{4\pi}\right]_0^a - \int_0^a \frac{x}{2} - \frac{a\sin\frac{2\pi}{a}x}{4\pi}\,dx + \int_0^a 4x\sin^2\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)dx\,\cos 3\omega t\right.$$

$$\left. + \left[\frac{x^2}{2} - \frac{xa\sin\frac{4\pi}{a}x}{8\pi}\right]_0^a - \int_0^a \frac{x}{2} - \frac{a\sin\frac{4\pi}{a}x}{8\pi}\,dx\right]$$

$\boxed{v = \frac{a\sin^3\frac{\pi}{a}x}{3\pi} \quad du = \sin^2\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)}$

$$= \frac{1}{a}\left[\frac{a^2}{2} - \left[\frac{x^2}{4} + \frac{a^2\cos\frac{2\pi}{a}x}{8\pi^2}\right]_0^a + 4\cos 3\omega t\left[\frac{xa\sin^3\frac{\pi}{a}x}{3\pi}\right]_0^a - \int_0^a \frac{a\sin^3\frac{\pi}{a}x}{3\pi}dx\right]$$

$$+ \frac{a^2}{2} - \left[\frac{x^2}{4} + \frac{a^2\cos\frac{4\pi}{a}x}{32\pi^2}\right]_0^a\Bigg]$$

$$= \frac{1}{a}\left[\frac{a^2}{2} - \frac{a^2}{4} - \frac{a^2}{8\pi^2} + \frac{a^2}{8\pi^2} - 4a^2\cos 3\omega t\left[\frac{\cos\left(\frac{3\pi}{a}x\right) - 9\cos\left(\frac{\pi}{a}x\right)}{36\pi^2}\right]_0^a + \frac{a^2}{2} - \frac{a^2}{4}\right]$$

$$= \frac{1}{a}\left[\frac{a^2}{4} - 4a^2\cos 3\omega t\left(\frac{-1+9-1+9}{36\pi^2}\right) + \frac{a^2}{4}\right]$$

$$= \frac{1}{a}\left[\frac{a^2}{4} - \frac{16a^2}{9\pi^2}\cos 3\omega t + \frac{a^2}{4}\right]$$

$$\boxed{= \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos 3\omega t\right]} \checkmark$$

Amplitude: $\frac{32}{9\pi^2}\frac{a}{2} = \boxed{0.3603\,\frac{a}{2}}$ ✓

frequency: $\boxed{\omega_f = 3\omega = \frac{3\pi^2\hbar}{2ma^2}}$

d) $$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = m\frac{a}{2}\left(\frac{32\omega}{3\pi^2}\sin 3\omega t\right)$$

$$\boxed{\langle p\rangle = \frac{16ma\omega}{3\pi^2}\sin 3\omega t} \checkmark$$

e) $c_1 = c_2$ so $P(E_1) = P(E_2)$ $\qquad \boxed{E_1 = \frac{\pi^2\hbar^2}{2ma^2}} \qquad \boxed{E_2 = \frac{4\pi^2\hbar^2}{2ma^2}}$

$$\langle H\rangle = E_1P(E_1) + E_2P(E_2)$$

$$= \frac{\pi^2\hbar^2}{4ma^2}(1+4)$$

$$\boxed{\langle H\rangle = \frac{5\pi^2\hbar^2}{4ma^2}} \checkmark$$

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2.6) $\boxed{\Psi(x,t) = \frac{1}{\sqrt{a}}\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \frac{1}{\sqrt{a}}e^{i\varphi}\sin\left(\frac{2\pi}{a}x\right)e^{-4i\omega t}}$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left(\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + e^{i\varphi}\sin\frac{2\pi}{a}x\, e^{-4i\omega t}\right)\left(\sin\left(\frac{\pi}{a}x\right)e^{i\omega t} + e^{-i\varphi}\sin\frac{2\pi}{a}x\, e^{4i\omega t}\right)$$

$$= \frac{1}{a}\left(\sin^2\frac{\pi}{a}x + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}\right)\left(e^{i\varphi}e^{-3i\omega t} + e^{-i\varphi}e^{3i\omega t}\right) + \sin^2\frac{2\pi}{a}x\right)$$

$$\boxed{|\Psi|^2 = \frac{1}{2a}\left(\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2\cos(3\omega t - \varphi)\right)}$$

$$\boxed{\langle x\rangle = \frac{a}{2}\left[1 - \frac{32}{9\pi^2}\cos(3\omega t - \varphi)\right]}$$ ✓

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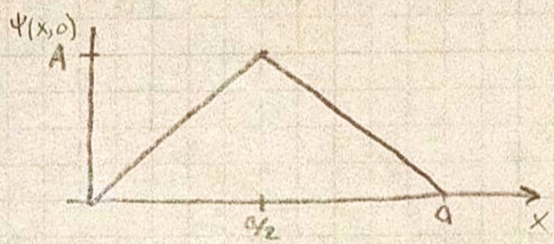
if $\varphi = \frac{\pi}{2}$, $\Psi(x,0) = A[\psi_1 + i\psi_2]$ $\quad\langle x\rangle = \frac{a}{2}\left(1 - \frac{32}{9\pi^2}\sin(3\omega t)\right)$ ✓

if $\varphi = \pi$, $\Psi(x,0) = A[\psi_1 - \psi_2]$ $\quad\langle x\rangle = \frac{a}{2}\left(1 + \frac{32}{9\pi^2}\cos(3\omega t)\right)$ ✓

2.7) a) $1 = \int_{-\infty}^{\infty}|\Psi|^2 dx = \int_0^{a/2}A^2x^2 + \int_{a/2}^{a}A^2(a-x)^2dx = A^2\left(\frac{1}{3}x^3\Big|_0^{a/2} + \frac{-(a-x)^3}{3}\Big|_{a/2}^{a}\right)$

$$A = \left(\frac{1}{\frac{a^3}{24} + \frac{a^3}{24}}\right)^{1/2}$$




$$\boxed{A = \frac{2\sqrt{3}}{a^{3/2}}}$$ ✓

b) $c_n = \frac{2\sqrt{6}}{a^2}\left[\int_0^{a/2}x\sin\left(\frac{n\pi}{a}x\right)dx + \int_{a/2}^{a}(a-x)\sin\left(\frac{n\pi}{a}x\right)dx\right]$ $\quad u = x \quad v = \frac{-a\cos\frac{n\pi}{a}x}{n\pi}$ $\quad du = dx \quad dv = \sin\left(\frac{n\pi}{a}x\right)$

$$= \frac{2\sqrt{6}}{a^2}\left[\frac{-ax\cos\frac{n\pi}{a}x}{n\pi}\Big|_0^{a/2} + \int_0^{a/2}\frac{a\cos\frac{n\pi}{a}x}{n\pi}dx - \frac{(a-x)a\cos\frac{n\pi}{a}x}{n\pi}\Big|_{a/2}^{a} - \int_{a/2}^{a}\frac{a\cos\frac{n\pi}{a}x}{n\pi}dx\right]$$

$$= \frac{2\sqrt{6}}{a^2}\left[\frac{-a^2\cos\frac{n\pi}{2}}{2n\pi} + \frac{a^2\sin\frac{n\pi}{a}x}{n^2\pi^2}\Big|_0^{a/2} + \frac{a^2\cos\frac{n\pi}{2}}{2n\pi} - \frac{a^2\sin\frac{n\pi}{a}x}{n^2\pi^2}\Big|_{a/2}^{a}\right]$$

$$= \frac{2\sqrt{6}}{a^2}\left[\frac{a^2\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{a^2\sin\frac{n\pi}{2}}{n^2\pi^2}\right]$$

$$= \frac{4\sqrt{6}\sin\frac{n\pi}{2}}{n^2\pi^2}$$ ✓

$$\boxed{= \begin{cases} 0 & \text{for } n \text{ even} \\ (-1)^{\frac{(n-1)}{2}}\frac{4\sqrt{6}}{n^2\pi^2} & \text{for } n \text{ odd}\end{cases}}$$

c) $P(E_1) = |c_1|^2 = \left(\frac{4\sqrt{6}}{\pi^2}\right)^2$

$$\boxed{P(E_1) = 0.9855}$$ ✓

d) $\langle H\rangle = \sum_{n\,odd}^{\infty}\left(\frac{4\sqrt{6}}{n^2\pi^2}\right)^2\frac{n^2\pi^2\hbar^2}{2ma^2}$

$$= \left(\frac{4\sqrt{6}\hbar}{\pi a}\right)^2\frac{1}{2m}\sum_{n\,odd}^{\infty}\frac{1}{n^2}$$

$$= \left(\frac{4\hbar}{\pi a}\right)^2\frac{3}{m}\frac{\pi^2}{8}$$

$$\boxed{= \frac{6\hbar^2}{a^2m}}$$ ✓

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**Problem 2.8** A particle of mass $m$ in the infinite square well (of width $a$) starts out in the state

$$\Psi(x,0) = \begin{cases} A, & 0 \leq x \leq a/2, \\ 0, & a/2 \leq x \leq a, \end{cases}$$

for some constant $A$, so it is (at $t = 0$) equally likely to be found at any point in the left half of the well. What is the probability that a measurement of the energy (at some later time $t$) would yield the value $\pi^2\hbar^2/2ma^2$?

**Problem 2.9** For the wave function in Example 2.2, find the expectation value of $H$, at time $t = 0$, the "old fashioned" way:

$$\langle H \rangle = \int \Psi(x,0)^* \, \hat{H} \Psi(x,0) \, dx.$$

Compare the result we got in Example 2.3. *Note:* Because $\langle H \rangle$ is independent of time, there is no loss of generality in using $t = 0$.

2.8) $1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = A^2 \int_0^{a/2} dx \rightarrow A = \sqrt{\frac{2}{a}}$ ✓

$c_n = \frac{2}{a} \int_0^{a/2} \sin\left(\frac{n\pi}{a} x\right) dx$

$= \left(\frac{2}{a}\right) \left[\frac{-a}{n\pi} \cos \frac{n\pi}{a} x\right]_0^{a/2}$

$= \left(\frac{2}{a}\right) \left(\frac{-a}{n\pi} \cos \frac{n\pi}{2} + \frac{a}{n\pi}\right)$

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$= \frac{2}{\pi}$ for $n = 1$

$|c_1|^2 = \frac{4}{\pi^2} = \boxed{0.4053 = P(E_1)}$ ✓

2.9) $\Psi(x,0) = Ax(a-x)$ $\hat{H} = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)$

$\langle H \rangle = \int \Psi(x,0)^* \hat{H} \Psi(x,0)\, dx$

$= \int_0^a Ax(a-x)\left(\frac{-A\hbar^2}{2m} \frac{\partial}{\partial x}\left((a-x) - x\right)\right)$

$= \frac{A^2\hbar^2}{2m} \int_0^a 2x(a-x)\, dx$

$= \frac{A^2\hbar^2}{m} \left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a$

$= \frac{A^2\hbar^2}{m} \left(\frac{a^3}{2} - \frac{a^3}{3}\right)$

$= \frac{A^2 a^3 \hbar^2}{6m}$ $A = \sqrt{\frac{30}{a^5}}$

$\boxed{\langle H \rangle = \frac{5\hbar^2}{a^2 m}}$ ✓ 15