Physics 531 Homework 10

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1. *Coupled spins.* Spin-1/2 particles A and B evolve under the influence of the following Hamiltonian (for simplicity take $\hbar = 1$ so that energies are expressed in frequency units):

$$H = -\Delta s_z^A - \Delta s_z^B + 4g\,\mathbf{s^A} \cdot \mathbf{s^B}$$

We work in the uncoupled basis $|ab\rangle \equiv |a\rangle \otimes |b\rangle$, where $a, b \in 0, 1$ and where states $|0\rangle$ $(|1\rangle)$ correspond to single spins aligned (antialigned) with the z-direction. As we discussed in lecture, the eigenstates of the Hamiltonian are $|00\rangle$, $|11\rangle$, and $2^{-1/2}(|01\rangle \pm |10\rangle)$.

(a) We prepare the initial state $|\psi(t = 0)\rangle = |01\rangle$. Since this state is not an eigenstate of the Hamiltonian, it will evolve in time. Write down the Hamiltonian matrix for this system in the subspace spanned by the states $|01\rangle$ and $|10\rangle$. Calculate the time evolved state $|\psi(t)\rangle$.

(b) Calculate the 4×4 density matrix $\hat{\rho}(t)$ corresponding to the state you found in part (a).

(c) Trace over the degrees of freedom of spin B to calculate the reduced density matrix describing spin A: $\hat{\rho}_A = \mathrm{Tr}_B(\hat{\rho})$.

(d) Calculate the three components of the Bloch vector $\mathbf{P_A}$ corresponding to $\hat{\rho}$ as a function of time. Again, the Bloch vector of an ensemble is defined from the following equation:

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \mathbf{P} \cdot \boldsymbol{\sigma})$$

where $\boldsymbol{\sigma}$ are the usual Pauli matrices.

Solution:

(a) Taking the Hamiltonian $H = -\Delta s_z^A - \Delta s_z^B + 4g\,\mathbf{s^A} \cdot \mathbf{s^B}$, we can rewrite this in matrix form.

$$\hat{H} = \begin{pmatrix} -\Delta + 2g & 0 & 0 & 0 \\ 0 & 0 & 2g & 0 \\ 0 & 2g & 0 & 0 \\ 0 & 0 & 0 & \Delta + 2g \end{pmatrix}$$

Where the rows and columns correspond (from top to bottom and left to right respectively) to the basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. The hamiltonian for the subspace spanned by $|01\rangle, |10\rangle$ is then

$$\hat{H} = \begin{pmatrix} 0 & 2g \\ 2g & 0 \end{pmatrix}$$

Our unitary operator is

$$
\begin{aligned}
U &= e^{-i\hat{H}t} \\
&= e^{-i2gt\sigma_x} \\
&= \cos{(2gt)}\mathbb{1} - i\sin{(2gt)}\sigma_x \\
&= \begin{pmatrix} c & -is \\ -is & c \end{pmatrix}
\end{aligned}
$$

In the basis $|01\rangle, |10\rangle$, our initial state $|01\rangle$ becomes

$$
|01\rangle \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}
$$

So then

$$
\begin{aligned}
|\psi(t)\rangle &= U\psi \\
&= \begin{pmatrix} c & -is \\ -is & c \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} c \\ -is \end{pmatrix} \\
&= \begin{pmatrix} \cos{(2gt)} \\ -i\sin{(2gt)} \end{pmatrix}
\end{aligned}
$$

Which in the original basis is

$$
\boxed{|\psi(t)\rangle = \begin{pmatrix} 0 \\ \cos{(2gt)} \\ -i\sin{(2gt)} \\ 0 \end{pmatrix}}
$$

(b) We can compute the density matrix from its definition.

$$
\hat{\rho} = |\psi\rangle\langle\psi| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c^2 & cs & 0 \\ 0 & -cs & s^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

As a check, notice that $\text{Tr}(\rho^2) = 1$. $\checkmark$

(c) Reducing the density matrix to only describe spin A, we get

$$
\hat{\rho}_A = \text{Tr}_B(\hat{\rho}) = \begin{pmatrix} c^2 & 0 \\ 0 & s^2 \end{pmatrix} = \begin{pmatrix} \cos^2{(2gt)} & 0 \\ 0 & \sin^2{(2gt)} \end{pmatrix}
$$

(d) We take the definition of the Bloch vector and set it equal to the density matrix from part c.

$$
\begin{aligned}
\begin{pmatrix} c^2 & 0 \\ 0 & s^2 \end{pmatrix} = \hat{\rho} &= \frac{1}{2}(\mathbb{1} + \mathbf{P}\cdot\boldsymbol{\sigma}) \\
&= \frac{1}{2}\begin{pmatrix} 1 + P_z & P_x - iP_y \\ P_x + iP_y & 1 - P_z \end{pmatrix}
\end{aligned}
$$

This is just a system of three equations that we must solve.

$$\begin{aligned} P_x + iP_y &= 0 \\ P_x - iPy &= 0 \\ P_z &= 2\cos^2(2gt) - 1 \end{aligned}$$

$$\begin{aligned} P_x &= 0 \\ P_x &= 0 \\ P_z &= 2\cos^2(2gt) - 1 \end{aligned}$$

We can check that $1 - P_z = 2 - \cos^2(2gt) = 2\sin^2(2gt)$. $\surd$

So therefore

$$\boxed{\vec{P} = (0, 0, 2\cos^2(2gt) - 1)}$$

(e)

2. Consider a system consisting of two spin-1/2 particles (a.k.a. "qubits"). For this problem I'll use the compact notation $X \equiv \sigma_x, Z \equiv \sigma_z, XX \equiv \sigma_x \otimes \sigma_x$, and $ZZ \equiv \sigma_z \otimes \sigma_z$.

(a) Construct the 4×4 matrices representing XX and ZZ in the basis spanned by the states $|00\rangle$, $|01\rangle$, and $|11\rangle$.

(b) Evaluate the commutator $[XX, ZZ]$.

(c) Consider the following Bell states, which are maximally entangled two qubit states:

$$\begin{aligned} |\Phi_\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \\ |\Psi_\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle), \end{aligned}$$

Show that these states are simultaneous eigenstates of the operators XX and ZZ. For each of the four Bell states, determine the eigenvalues of XX and ZZ.

Solution:

(a) We have the matrices

$$XX = \sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$ZZ = \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) The commutator [XX, ZZ] then becomes

$$[XX, ZZ] = XX\ ZZ - ZZ\ XX$$
$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boxed{[XX, ZZ] = \overleftrightarrow{0}}$$

(c) The eigenstates of the XX and ZZ matrices are

$$XX\,|\Phi_+\rangle = \frac{1}{\sqrt{2}} XX \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \checkmark$$

$$XX\,|\Phi_-\rangle = \frac{1}{\sqrt{2}} XX \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \checkmark$$

$$XX\,|\Psi_+\rangle = \frac{1}{\sqrt{2}} XX \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \checkmark$$

$$XX\,|\Psi_-\rangle = \frac{1}{\sqrt{2}} XX \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \checkmark$$

$$ZZ\,|\Phi_+\rangle = \frac{1}{\sqrt{2}} ZZ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \checkmark$$

$$ZZ\,|\Phi_-\rangle = \frac{1}{\sqrt{2}} ZZ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \quad \checkmark$$

$$ZZ\,|\Psi_+\rangle = \frac{1}{\sqrt{2}} ZZ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ -1 \\ 0 \end{pmatrix} \quad \checkmark$$

$$ZZ\,|\Psi_-\rangle = \frac{1}{\sqrt{2}} ZZ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \checkmark$$

which correspond to the XX eigenvalues $\lambda_{\Phi_+}, \lambda_{\Psi_+} = 1, \lambda_{\Phi_-}, \lambda_{\Psi_-} = -1$ and ZZ eigenvalues of $\lambda_{\Phi_+}, \lambda_{\Phi_-} = 1, \lambda_{\Psi_+}, \lambda_{\Psi_-} = -1$

(d) If we prepare the state

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

the only possible outcomes of XX are its eigenvalues of ± 1. If XX changes from 1 to -1, we know that we are either in the state

$$|\Phi_-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

or

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

In the $|\Phi_-\rangle$ case, there is no bit flip, but there is a phase flip. In the $|\Psi_-\rangle$ case, there is a bit flip on B and a phase flip.

So, we know that there was a phase flip and bit bit flip on A, but we aren't sure if there was a bit flip on B.

(e) If ZZ goes tom 1 to -1, the possible states are

$$|\Phi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and

$$|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

this tells us that qubit B had a bit flip and A did not, but we're unsure whether a phase flip occurred. However, the qubits are indistinguishable,

so the most that we can conclude is that a bit flip occurred on one of the bits.