Math 632 Spring 2020 Homework 9

Due Thursday, April 30, 4 PM

Homework is to be handed in on Canvas by uploading a PDF file.

Please check the homework instructions on the course homepage. In particular, remember to observe rules of academic integrity. You are encouraged to discuss the problems with fellow students, but copied work is not acceptable and will result in zero credit. One can find solutions to many exercises on the web. However, the point of the homework is to give you the problem solving practice you will need in the exams. Hence it is not wise to take shortcuts to secure the few points that come from homework.

Even when a problem seems easy, be sure to give some justification for your answer. Otherwise your solution cannot get full credit.

1. This problem is for basic practice with exponential distributions. Let $U \sim \text{Exp}(\alpha)$ and $V \sim \text{Exp}(\beta)$ be independent. Calculate the expectation $E[U \cdot I_{\{U<V\}}]$. Here $I_{\{U<V\}}$ is the indicator random variable of the event $U < V$.

Hint: If you are at a loss at how to proceed, then just remember that if f is the joint probability density function of a pair of random variables (X, Y), then for any function h,

$$E[\,h(X,Y)\,] = \iint_{\mathbb{R}^2} h(x,y)\, f(x,y)\, dx\, dy.$$

There are also quicker ways for the task at hand, for example by using exponential races or by conditioning. The answer is $\frac{\alpha}{(\alpha+\beta)^2}$.

2. Consider the CTMC with state space be $\mathcal{S} = \{1, 2, 3\}$ and jump rates

$$q(1,2) = 2, \quad q(1,3) = q(3,2) = 3, \quad q(2,1) = 5, \quad q(2,3) = 1.$$

(The unmentioned jump rate $q(3,1)$ is zero, so a jump from 3 to 1 is not possible.) Let $S = \min\{t \geq 0 : X_t \neq 1\}$ be the first time when the process is no longer in state 1. Let $T = \min\{t \geq 0 : X_t = 2\}$ be the first time the process is in state 2.

(a) Draw the arrow diagram labeled with the rates.

(b) Find the holding parameters and the routing matrix.

(c) Find the probability $P_1(s \leq S \leq t)$ for $t > s > 0$.

(d) Calculate $E_1[T]$. (The answer is $\frac{2}{5}$.)
Hint: There are two distinct approaches. For an ideal solution, do both.
(1) Use the jump chain and the holding times.
(2) Use the Poisson clocks attached to the arrows of the diagram. Let $T_{xy} \sim \text{Exp}(q(x,y))$ denote an exponential random variable associated to the arrow from x to y. If $T_{12} < T_{13}$ then the process jumps from 1 to 2 and $T = T_{12}$. What is T equal to in the opposite case $T_{12} > T_{13}$? Formulate an expression for T that allows you to take advantage of the calculation in problem 1.

3. A machine functions for an exponentially distributed amount of time with rate λ before it fails. When it fails, the failure is one of three types labeled $i = 1, 2, 3$ with probabilities $1/2, 1/3, 1/6$, respectively. Suppose a failure of type i takes an exponential amount of time with rate μ_i to repair. Formulate a CTMC model for the state of the machine with state space $\{0, 1, 2, 3\}$ where state 0 means that the machine is functional and state $i \in \{1, 2, 3\}$ means that the machine is under repair for a type i failure.

(a) Give the routing matrix and jump rates of the Markov chain.

(b) Find the stationary distribution.

(c) We come to observe a functioning machine. What is the probability that the next two failures are of the same type?

4. There are two tennis courts in a gym. Pairs of players arrive at rate 2 per hour. Each pair plays for an exponentially distributed amount of time with mean 1 hour. If both courts are occupied, an arriving pair waits for a free court. If there is already one pair of players waiting, new arrivals will leave. A waiting pair of players takes a court as soon as one becomes free. However, a waiting pair of players waits only for an exponentially distributed amount of time with **mean** $\frac{1}{6}$ hour and then leaves if they cannot get on a court.

(a) Give the generator matrix of the Markov model with state space $\{0, 1, 2, 3\}$ that keeps track of the number of pairs of players in the system.

(b) Every pair of players pays a rate of \$2 dollars per hour while using the court. What is the long term rate of revenue of the tennis courts, in dollars per hour?

(c) Assuming that the system has been running for a very long time, what is the probability that an arriving pair of players finds immediately a free court? What is the probability that this pair gets to play at all? (The answer to the last question is $\frac{7}{11}$.)

5. Consider a two-station queueing network. Arrivals occur only at the first server and do so at rate 2. If an arriving customer finds server 1 free he enters the system; otherwise he goes away. When a customer is done at the first server he moves on to the second server if it is free and leaves the system if it is not. A customer who has completed service at the second server leaves the system. Suppose that server 1 serves at rate 1 while server 2 serves at rate 3.

(a) Formulate a Markov chain model for this system with state space $\{0, 1, 2, 12\}$ where the state indicates the servers who are busy.

For parts (b) *and* (c) *below we do not expect rigorous solutions. Use common sense to arrive at the correct numerical answers and explain your reasoning.*

(b) In the long run what proportion of the arriving customers enter the system?

(c) In the long run what proportion of the arriving customers visit server 2?
Hint: Think of the possible states in which an arriving customer finds the system and the probability that they reach server 2.

6. Consider an $M/M/\infty$ queueing system with infinitely many servers (see Durrett Example 4.17 on page 168) where customers arrive at rate λ, and the service time of each customer is a rate μ exponential random variable. Let X_t denote the number of customers in the system at time t. Assume that $X_0 = 0$.

(a) Write down explicitly the Kolmogorov forward equations for this process.

(b) Set $m(t) = E_0[X_t]$. Prove that

$$m'(t) = \lambda - \mu m(t).$$

Hint: Use the equations from (a). Feel free to differentiate the series term by term.

(c) Solve the differential equation for $m(t)$.
Hint: Find $\frac{d}{dt}(m(t)e^{\mu t})$ first.

(d) Evaluate $\lim_{t\to\infty} m(t)$. The stationary distribution for X_t is given in Example 4.17 of Durrett's book. Compare the limit you found to the expected value of the stationary distribution.

Math 632 Homework 4 Luis Guzman

1) $$E[U \cdot I_{\{u<v\}}] = \int_0^\infty \int_u^\infty u P(U=u) P(V=v) \, dv \, du$$

$$= \int_0^\infty u \alpha e^{-\alpha u} \left[0 + e^{-\beta u} \right] du$$

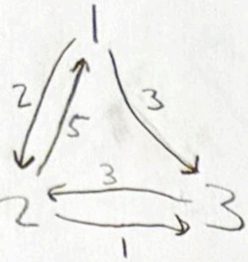
$$= \int_0^\infty u \alpha e^{-(\alpha+\beta)u} du$$

$$= \alpha \left[\left(\frac{-u}{\alpha+\beta} - \frac{1}{(\alpha+\beta)^2} \right) e^{-(\alpha+\beta)u} \right]_0^\infty$$

$$= \alpha \left(0 + \frac{1}{(\alpha+\beta)^2} \right)$$

$$\boxed{= \frac{\alpha}{(\alpha+\beta)^2}}$$

2) a)



b) holding parameters: $\lambda(x) = \sum_y q(x,y)$

$$\lambda(1) = q(1,2) + q(1,3) = 2+3 = 5$$
$$\lambda(2) = q(2,1) + q(2,3) = 5+1 = 6$$
$$\lambda(3) = q(3,1) + q(3,2) = 0+3 = 3$$

$$\boxed{\begin{aligned}\lambda(1)&=5\\ \lambda(2)&=6\\ \lambda(3)&=3\end{aligned}}$$

routing matrix: $r(x,y) = \dfrac{q(x,y)}{\lambda(x)}$

$$\boxed{R = \begin{bmatrix} 0 & 2/5 & 3/5 \\ 5/6 & 0 & 1/6 \\ 0 & 1 & 0 \end{bmatrix}}$$

c) $P_1(s \le S \le t) = P(s \le J_1 \le t)$ where J_1 is the holding time in state 1

$= F_{J_1}(t) - F_{J_2}(s)$ rewrite in terms of CDF

$= 1 - e^{-\lambda(1)t} - 1 + e^{-\lambda(1)s}$

$$\boxed{P(s \le S \le t) = e^{-5s} - e^{-5t}}$$

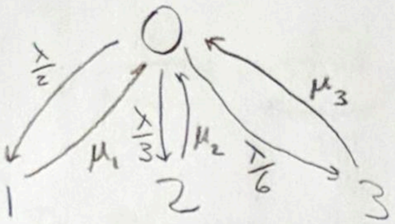
d) $E_1[T] = \underbrace{P(T_{12} < T_{13}) E[J_1]}_{\text{go directly to two}} + \underbrace{P(T_{13} < T_{12}) E[J_1 + J_2]}_{\text{go to 3 first}}$

$= r(1,2) E[J_1] + r(1,3)(E[J_1] + E[J_2])$

$= \frac{2}{5} \cdot \frac{1}{5} + \frac{3}{5}\left(\frac{1}{5} + \frac{1}{3}\right)$

$$\boxed{E_1[T] = \frac{2}{5}}$$

3)



a) jump rates:

$$q(x,y) = \lambda(x)\, r(x,y)$$

$r(i,0) = 1$ for $i = 1,2,3$ so $\boxed{q(i,0) = \mu_i}$

$$\boxed{\begin{aligned} q(0,1) &= \lambda\, r(0,1) = \frac{\lambda}{2} \\ q(0,2) &= \frac{\lambda}{3} \\ q(0,3) &= \frac{\lambda}{6} \end{aligned}}$$

$$\boxed{R = \begin{bmatrix} 0 & 1/2 & 1/3 & 1/6 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}$$

Check: exponential race $P(K=1) = \frac{\lambda_1}{\sum_i \lambda_i} = \frac{\lambda/2}{\lambda} = \frac{1}{2}$ ✓

b)

$$Q = \begin{bmatrix} -1 & 1/2 & 1/3 & 1/6 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Stationary distribution: $\pi Q = 0$

$$\begin{cases} \pi(0) = \pi(1) + \pi(2) + \pi(3) \\ \pi(1) = \frac{1}{2}\pi(0) \\ \pi(2) = \frac{1}{3}\pi(0) \\ \pi(3) = \frac{1}{6}\pi(0) \end{cases}$$

$$1 = \sum_{k=0}^{3} \pi(k) = 2\pi(0) \Rightarrow \pi(0) = \frac{1}{2}$$

$$\boxed{\pi = \left[\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{6} \quad \frac{1}{12}\right]}$$

c) P_0(Next two failures are the same)

$$= P_0(T_{01} < T_{02} \vee T_{03})^2 + P(T_{02} < T_{01} \vee T_{03})^2 + P(T_{03} < T_{01} \vee T_{02})^2$$

$$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{6}\right)^2$$

$$\boxed{= \frac{7}{18}}$$

4)

$$\begin{tikzcd} 0 \arrow[r, bend left, "2"] & 1 \arrow[l, bend left, "1"] \arrow[r, bend left, "2"] & 2 \arrow[l, bend left, "2"] \arrow[r, bend left, "2"] & 3 \arrow[l, bend left, "8"] \end{tikzcd}$$

$\lambda(0) = 2$
$\lambda(1) = 3$
$\lambda(2) = 3$
$\lambda(3) = 7$

The MC increases by a constant rate of 2/hr.
If no one is waiting, the MC decreases by a constant rate of 1/hr $\left(\frac{1}{\text{mean}}\right)$
If someone is waiting, there is an exponential race between the court becoming available and the waiters leaving. Due to the memoryless property, E[court becomes available] equals 1 no matter when they showed up. The winner of the exponential race then has distribution $\text{Exp}\left(\sum_{i=1}^{n} \lambda_i\right) = \text{Exp}(2+6)$

$$\boxed{Q = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 8 & -8 \end{bmatrix}}$$

$\pi Q = 0$

$$\begin{cases} 2\pi(0) = \pi(1) \\ 3\pi(1) = 2\pi(0) + 2\pi(2) \\ 4\pi(2) = 2\pi(1) + 8\pi(3) \\ 8\pi(3) = 2\pi(2) \end{cases} \Rightarrow \begin{array}{l} \pi(1) = 2\pi(0) \\ \pi(2) = 2\pi(0) \\ \pi(3) = \frac{1}{2}\pi(0) \end{array}$$

$1 = \pi(0)\left(1 + 2 + 2 + \frac{1}{2}\right)$

$\pi(0) = 2/11$

$\pi = \left[\frac{2}{11} \quad \frac{4}{11} \quad \frac{4}{11} \quad \frac{1}{11}\right]$

b)
$$f(X_s) = \begin{cases} 2X_s & \text{for } X_s = \{0, 1, 2\} \\ 4 & \text{for } X_s = 3 \quad \left(\text{waiting players don't pay}\right) \end{cases}$$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X_s)\, ds = \sum_x f(x)\pi(x) \quad \text{by SLLN}$$

$$= 0 \cdot \frac{2}{11} + 2 \cdot \frac{4}{11} + 4 \cdot \frac{4}{11} + 4 \cdot \frac{1}{11}$$

$$\boxed{= \frac{28}{11} \sim \$2.55/\text{hr}}$$

c) $P(\text{arriving pair finds free court}) = \lim_{t \to \infty} \frac{1}{t} \int_0^t I_{X_s = \{0,1\}}\, ds = \pi(0) + \pi(1) \boxed{= \frac{6}{11}}$

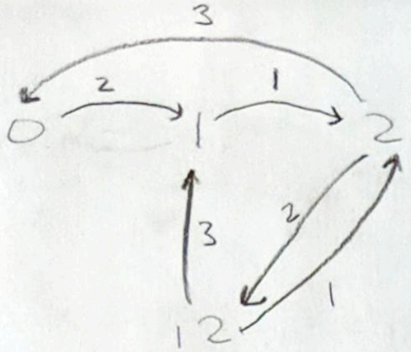
$P(\text{arriving pair gets to play}) = P(\text{arriving pair finds free court}) + \underbrace{P(\text{court frees up before they leave})}_{\to \text{ exponential race}} P(\text{MC is in state 2})$

$$= \frac{6}{11} + \frac{1}{4}\frac{4}{11}$$

$P(K=1) = \frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{2}{8} = \frac{1}{4}$

$$\boxed{= \frac{7}{11}}$$

5) a)



$$Q = \begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 3 & 0 & -5 & 2 \\ 0 & 3 & 1 & -4 \end{bmatrix}$$

$$\begin{cases} 2\pi(0) = 3\pi(2) \\ \pi(1) = 2\pi(0) + 3\pi(12) \\ 5\pi(2) = \pi(1) + \pi(12) \\ 4\pi(12) = 2\pi(2) \end{cases} \Rightarrow \begin{aligned} &\pi(1) = 3\pi(0) \\ &\pi(2) = \tfrac{2}{3}\pi(0) \\ &\pi(12) = \tfrac{1}{3}\pi(0) \\ &1 = \textstyle\sum \pi(k) = \pi(0)\left(1 + 3 + \tfrac{2}{3} + \tfrac{1}{3}\right) \\ &\pi(0) = \tfrac{1}{5} \end{aligned}$$

$$\pi = \begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{2}{15} & \frac{1}{15} \end{bmatrix}$$

b) proportion of customers that enter the system is the same as the proportion of time we spent with server 1 available

$$P(\text{customer enters system}) = \pi(0) + \pi(2)$$
$$= \frac{1}{5} + \frac{2}{15}$$
$$\boxed{= \frac{1}{3}}$$

c) $P(\underset{\text{customer at 1}}{\text{doesn't make it to 2}}) = \frac{1}{1+3} = \frac{1}{4}$ exponential race

$$P(\text{customer visits 2}) = P(\text{customer makes it into system})\big(1 - P(\text{don't make it to 2})\big)$$
$$= \frac{1}{3}\left(1 - \frac{1}{4}\right)$$
$$\boxed{= \frac{1}{4}}$$

6)
$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] \arrow[r, bend left, "\lambda"] & \cdots \arrow[l, bend left, "4\mu"] \end{tikzcd}$$

a)
$$p_t'(k, k+1) = \sum_z p_t(k, z)q(z, k+1)$$
$$= p_t(k, k)q(k, k+1) + p_t(k, k+2)q(k+2, k+1)$$

$$\boxed{\begin{aligned} p_t'(k, k+1) &= p_t(k, k)\lambda + p_t(k, k+2)\cdot(k+2)\mu \\ p_t'(k+1, k) &= p_t(k+1, k+1)(k+1)\cdot\mu + p_t(k+1, k-1)\lambda \end{aligned}}$$
$$\mathbb{P}_0 = I$$

$$p_t'(1, 0) = p_t(1,1)q(1,0) = p_t(1,1)\mu$$

b)
$$m(t) = E_0[X_t]$$
$$m'(t) = \frac{d}{dt}E_0[X_t]$$