Math 632 Spring 2020 Homework 6

Due Sunday, March 29, 4 PM

Note that problem 5(c) is a little different from the rest: you need to combine ideas in a way that goes beyond what we have done thus far in the course.

1. Let X and Y be discrete random variables with joint probability mass function

$$f(a,b) = P(X=a, Y=b) = \begin{cases} a^b/20 & \text{if } a \in \{1,2,3\}, b \in \{1,2\}, \\ 0 & \text{if } (a,b) \notin \{1,2,3\} \times \{1,2\}. \end{cases}$$

(a) Find the function $H(y) = E[X|Y=y]$ for the values y such that $P(Y=y) > 0$.

(b) Compute the probability mass function of the random variable $Z = E[X|Y]$.

2. Let $X_1, X_2, \ldots$ be a sequence of discrete bounded i.i.d. random variables. Let $S_n = X_1 + \cdots + X_n$, $n \geq 1$.

(a) Use the definition of conditional expectation to verify that $E[X_i|S_n] = E[X_1|S_n]$ for all $i = 1, \ldots, n$ and $n \geq 1$.
Hint: Start by showing that $P(X_i = x|S_n = y) = P(X_1 = x|S_n = y)$. The observation that $S_n - X_i$ and $X_2 + \cdots + X_n$ have the same distribution can be useful.

(b) Use the above result to compute $E[X_1|S_n]$. Then find a formula for $E[S_m|S_n]$ for all $1 \leq m \leq n$.

3. Let $Y_0, Y_1, \ldots$ be i.i.d. random variables with possible values $\{-1, 2\}$ and distribution

$$P(Y_1 = 2) = \frac{1}{3}, \quad P(Y_1 = -1) = \frac{2}{3}.$$

Which of the following sequences are martingales with respect to $\{Y_k : k \geq 0\}$? Explain why.

(a) $M_n = Y_0^n$, $n \geq 0$;

(b) $V_n = \prod_{k=0}^{n}(1 + Y_k)$, $n \geq 0$;

(c) $W_n = Y_n$, $n \geq 0$;

4. Suppose that $\{X_k, k \geq 1\}$ is a sequence of i.i.d. random variables with $P(X_1 = \pm 1) = \frac{1}{2}$. Let $S_n = \sum_{k=1}^{n} X_k$ (i.e. $S_n, n \geq 1$ is a symmetric simple random walk with steps $X_k, k \geq 1$).

(a) Compute $E[S_{n+1}^3|X_1, \ldots, X_n]$ for $n \geq 1$.
Hint: Check out Example 3.8 in the lecture notes (Version Mar/04/2020) for inspiration.

(b) Find deterministic coefficients a_n, b_n, c_n possibly depending on n so that $M_n = S_n^3 + a_n S_n^2 + b_n S_n + c_n$ is a martingale with respect to $\{X_k, k \geq 1\}$.

5. Let $\{X_k\}_{k\geq 1}$ be i.i.d. random variables such that $X_k > 0$, $E[X_k] = 1$ and $E[\log X_k] < 0$.

(a) Give an example of a random variable X_k that satisfies the assumptions above and has exactly two values.

(b) Give a justification for why the limit

$$Z = \lim_{n\to\infty} \prod_{k=1}^{n} X_k$$

exists with probability one.

(c) Find the exact value of Z. Hint: Applying the SLLN to the random variables $\{\log X_k\}_{k\geq 1}$ can give you useful information.

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1) a) $H(y) = E[X|Y=y] = \sum_k k\, P(X=k|Y=y)$

$P(Y=y) = \sum_x P(X=x, Y=y)$

$P(Y=1) = \frac{1}{20} + \frac{2}{20} + \frac{3}{20} = \frac{3}{10}$

$P(Y=2) = \frac{1}{20} + \frac{2^2}{20} + \frac{3^2}{20} = \frac{7}{10}$

$= \sum_k k \frac{P(X=k, Y=y)}{P(Y=y)}$

$= \frac{1}{P(Y=y)}\left(1\cdot P(X=1, Y=y) + 2\cdot P(X=2, Y=y) + 3\cdot P(X=3, Y=y)\right)$

$= \frac{1}{P(Y=y)}\left(\frac{1}{20} + 2\cdot\frac{2^y}{20} + 3\cdot\frac{3^y}{20}\right)$

$= \frac{1}{20\,P(Y=y)}\left(1 + 2^{y+1} + 3^{y+1}\right) = \frac{1+2^{y+1}+3^{y+1}}{20\left(-\frac{3}{10}(y-2) + \frac{7}{10}(y-1)\right)}$

$$\boxed{H(y) = \begin{cases} \frac{7}{3} & \text{for } y=1 \\ \frac{18}{7} & \text{for } y=2 \end{cases}}$$

b) $Z = E[X|Y]$

$= \frac{1 + 2^{Y+1} + 3^{Y+1}}{20\left(-\frac{3}{10}(Y-2) + \frac{7}{10}(Y-1)\right)}$

z	$P(Z=z)$
$7/3$	$3/10$
$18/7$	$7/10$

Check: $E[Z] = \frac{7}{3}\frac{3}{10} + \frac{18}{7}\cdot\frac{7}{10} = 2.5$ ✓

$E[X] = \frac{1}{10} + \frac{6}{10} + \frac{18}{10} = 2.5$

$P(X=1) = \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$

$P(X=2) = \frac{2}{20} + \frac{4}{20} = \frac{3}{10}$

$P(X=3) = \frac{3}{20} + \frac{9}{20} = \frac{6}{10}$

2) a) $E[X_i | S_n] = E[S_n | S_n] - E[S_n - X_i | S_n]$ by linearity

$= E[S_n | S_n] - E[X_2 + \cdots + X_n | S_n]$ since $S_n - X_i$ and $X_2 + \cdots + X_n$ are identically distributed

$= E[S_n - X_2 - \cdots - X_n | S_n]$ by linearity

$$\boxed{E[X_i | S_n] = E[X_1 | S_n]}$$

b) part a tells us that $E[X_1 | S_n] = \cdots = E[X_n | S_n]$

We also have $E[S_n | S_n] = E[X_1 | S_n] + \cdots + E[X_n | S_n]$ by linearity

$\Rightarrow \quad S_n = n E[X_1 | S_n]$

$\Rightarrow \boxed{E[X_1 | S_n] = \frac{S_n}{n}}$

$E[S_m | S_n] = E[X_1 + \cdots + X_m | S_n]$

$= E[X_1 | S_n] + \cdots + E[X_m | S_n]$

$= m E[X_1 | S_n]$

$$\boxed{E[S_m | S_n] = \frac{m}{n} S_n}$$

3) a) $M_n = f(Y_0)$ ✓

$E[M_n] = E[Y_0^n] < \infty$ ✓ First two properties hold.

$E[M_{n+1} | Y_0, \cdots Y_n] = E[M_{n+1} | Y_0]$ since M_n is independent of Y_n $\forall n > 0$

$= M_{n+1}$ since M_{n+1} is a function of Y_0

$= Y_0^{n+1} \neq M_n \longrightarrow$ $\boxed{M_n \text{ is not a Martingale}}$

b) By inspection, first two properties hold. ✓

$E[V_{n+1} | Y_0 \cdots Y_n] = \frac{1}{3}(1+2)\prod_{K=0}^{n}(1+Y_K) + \cancel{\frac{2}{3}(1-1)\prod_{K=0}^{n}(1+Y_K)}^{\to 0}$

$= \prod_{K=0}^{n}(1+Y_K)$

$= V_n \longrightarrow$ $\boxed{V_n \text{ is a Martingale}}$

c) By inspection, first two properties hold. ✓

$E[W_{n+1} | Y_0 \cdots Y_n] = E[W_{n+1}]$ since W_{n+1} is independent of $(Y_0, \cdots Y_n)$

$= E[Y_{n+1}]$

$= E[Y_n]$ since Y_i's are iid

$\neq W_n$ since W_n is random and $E[Y_n]$ is a constant.

$\rightarrow$ $\boxed{W_n \text{ is not a martingale}}$

4) a) $E[S_{n+1}^3 | X_1, \dots, X_n] = E[(S_n + X_{n+1})^3 | X_1, \dots X_n]$

$$= E[S_n^3 | \vec{X}_n] + E[3S_n^2 X_{n+1} | \vec{X}_n] + E[3S_n X_{n+1}^2 | \vec{X}_n] + E[X_{n+1}^3 | \vec{X}_n] \quad \text{by linearity}$$

$$= S_n^3 + 3S_n^2 \cancelto{0}{E[X_{n+1}]} + 3S_n E[X_{n+1}^2] + \cancelto{0}{E[X_{n+1}^3]}$$

$$= S_n^3 + 3S_n$$

$$\boxed{E[S_{n+1}^3 | X_1, \dots X_n] = S_n(S_n^2 + 3)}$$

b) $E[M_{n+1} | X_1, \dots X_n] = M_n$

LHS: $E[M_{n+1} | X_1, \dots X_n] = E[S_{n+1}^3 | \vec{X}_n] + E[a_{n+1} S_{n+1}^2 | \vec{X}_n] + E[b_{n+1} S_{n+1} | \vec{X}_n] + c_{n+1}$

$$= S_n(S_n^2 + 3) + a_{n+1}(S_n^2 + 1) + b_{n+1} S_n + c_{n+1}$$ using result from 4a and example 3.8

Set equal to RHS

$$S_n(S_n^2 + 3) + a_{n+1}(S_n^2 + 1) + b_{n+1} S_n + c_{n+1} = S_n^3 + a_n S_n^2 + b_n S_n + c_n$$

$$a_{n+1} S_n^2 + (b_{n+1} + 3) S_n + (c_{n+1} + a_{n+1}) = a_n S_n^2 + b_n S_n + c_n$$

Matching coefficients, we get:

$$a_{n+1} = a_n \qquad b_{n+1} = b_n - 3 \qquad c_{n+1} = c_n - a_{n+1}$$

We can then pick starting a_0, b_0, c_0 values and

$$\boxed{a_n = a_0 \qquad b_n = b_0 - 3n \qquad c_n = c_0 - n a_0}$$

5) a)

X	$P(X=x)$
0.1	1/2
1.9	1/2

$E[X] = \frac{0.1}{2} + \frac{1.9}{2} = 1$ ✓

$E[\ln X] = \frac{\ln 0.1}{2} + \frac{\ln 1.9}{2} = -0.83$ ✓

b) Is $M_n = \prod_{k=1}^{n} X_k$ a martingale? First two properties hold ✓

$$E[M_{n+1} | \vec{X}_n] = \frac{1}{2} \cdot 0.1\, M_n + \frac{1}{2} \cdot 1.9\, M_n$$

$$= M_n$$

M_n is also bounded below by 0 since $X_k > 0$
by the Martingale limit theorem, the limit

$$Z = \lim_{n\to\infty} \prod_{k=1}^{n} X_k \quad \text{exists} \quad \text{w.p. } 1$$

c) $Z = \lim_{n\to\infty} \prod_{k=1}^{n} X_k$

$= \lim_{n\to\infty} X_1^{n-m} X_2^{m}$ Split into two possibilities /

$$\ln Z = \ln \lim_{n\to\infty} \prod_{k=1}^{n} X_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \ln(X_k) \quad \text{by log-product rule}$$

$$= \lim_{n\to\infty} \underbrace{\frac{1}{n} \sum_{k=1}^{n} \ln(X_k^n)}_{\text{call this a new R.V. } S_n}$$

$$= \lim_{n\to\infty} E[\ln(X_k^n)] \quad \text{by SLLN}$$

$$= \lim_{n\to\infty} \frac{1}{2} \ln(0.1^n) + \frac{1}{2} \ln(1.9^n)$$

$$= \lim_{n\to\infty} \ln((0.1 \cdot 1.9)^{n/2})$$

$$Z = \lim_{n\to\infty} (0.19^{n/2})$$

$$\boxed{Z = 0}$$