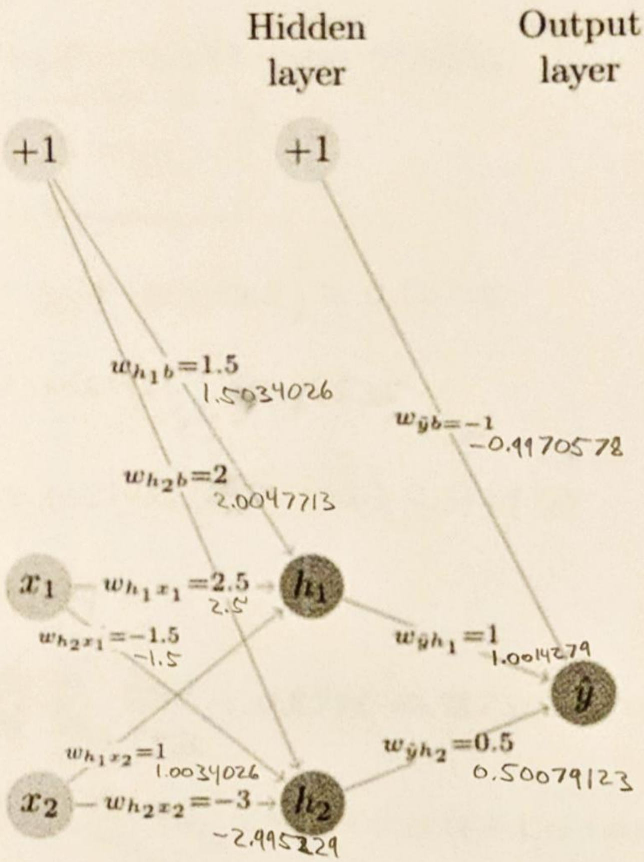
Problem 1. Neural Networks [15 points]

The figure below shows a 2-layer, feed-forward neural network with two hidden-layer nodes and one output node. x_1 and x_2 are the two inputs. For the following questions, assume the learning rate is $\alpha = 0.1$. Each node also has a bias input value of +1. Assume there is a sigmoid activation function at the hidden layer nodes and at the output layer node. A sigmoid activation function takes the form: $g(z) = \frac{1}{1+e^{-z}}$ where $z = \sum_{i=1}^{n} w_i x_i$ and w_i is the i^{th} incoming weight to a node, x_i is the i^{th} incoming input value, and n is the number of incoming edges to the node.



(a) [5] Calculate the output values at nodes h_1, h_2 and $\hat{y}$ of this network for input $\{x_1 = 0, x_2 = 1\}$. Each unit produces as its output the real value computed by the unit's associated sigmoid function. Show all steps in your calculation.

(b) [10] Compute *one* (1) step of the **backpropagation algorithm** for a given example with input $\{x_1 = 0, x_2 = 1\}$ and target output $y = 1$. The network output is the real-valued output of the sigmoid function, so the error on the given example is defined as $E = \frac{1}{2}(y - O)^2$ where O is the real-valued network output of that example at the output node, and y is the integer-valued target output for that example. Compute the updated weights for both the hidden layer and the output layer (there are nine updated weights in total (i.e., the three incoming weights to node h_1, the three incoming weights to node h_2 and the three incoming weights to node $\hat{y}$) by performing ONE step of gradient descent. Show all steps in your calculation.

a) $X_1 = 0 \qquad X_2 = 1$

$$Z_1 = 2.5 \cdot 0 + 1 \cdot 1 + 1.5 \cdot 1 = 2.5$$

$$Z_2 = -1.5 \cdot 0 - 3 \cdot 1 + 2 \cdot 1 = -1$$

$$\boxed{h_1 = \frac{1}{1+e^{-2.5}} = 0.92414}$$

$$\boxed{h_2 = \frac{1}{1+e^{1}} = 0.26894}$$

$$Z_{out} = 1 \cdot 0.92414 + 0.5 \cdot 0.26894 - 1 \cdot 1 = 0.05861$$

$$\boxed{\hat{y} = \frac{1}{1+e^{-0.05861}} = 0.51465}$$

b) $E = \frac{1}{2}(y-o)^2 = \frac{1}{2}(1-0.51465)^2 = 0.11778$

$$\frac{\partial E}{\partial \hat{y}} = (y-o) = (1-0.51465) = 0.48535$$

$$\frac{\partial \hat{y}}{\partial z_{out}} = \hat{y}(1-\hat{y}) = 0.51465(1-0.51465) = 0.24979$$

$$\frac{\partial z_{out}}{\partial w_{\hat{y}h_1}} = h_1 = 0.92414$$

So then $\dfrac{\partial E}{\partial w_{\hat{y}h_1}} = \dfrac{\partial E}{\partial \hat{y}} \dfrac{\partial \hat{y}}{\partial z_{out}} \dfrac{\partial z_{out}}{\partial w_{\hat{y}h_1}} = 0.11778 \cdot 0.48535 \cdot 0.24979 = 0.014279$

$$W_{\hat{y}h_1} = \Delta W_{\hat{y}h_1} + W_{\hat{y}h_1} = \alpha \frac{\partial E}{\partial w_{\hat{y}h_1}} + W_{\hat{y}h_1} = 0.1(0.014279) + 1 \boxed{= 1.0014279}$$

Similarly,

$$W_{\hat{y}h_2} = \alpha h_2 \hat{y}(1-\hat{y})(y-o) + W_{\hat{y}h_2} = 0.1(0.26894)(0.24979)(0.11778) + 0.5 \boxed{= 0.50079123}$$

$$W_{\hat{y}b} = \alpha(1)\hat{y}(1-\hat{y})(y-o) + W_{\hat{y}b} = 0.1(1)(0.24979)(0.11778) - 1 \boxed{= -0.9970578}$$

Input-hidden weights

$$\Delta W_{h_1 x_1} = \alpha h_1 g'(z_1) \sum_k W_{h_1 \hat{y}} \Delta_k$$

$$= \alpha x_1 \cdot h_1(1-h_1) W_{\hat{y}h_1}(y-\hat{y})$$

$$= 0.1 \cdot 0 \cdot (0.92414)(0.07586) \cdot 1 (0.48535)$$

$$= 0$$

$$\boxed{W_{h_1 x_1} = 2.5}$$

Similarly $\boxed{W_{h_2 x_1} = -1.5}$

$$W_{h_1 x_2} = 0.1 \cdot 1 \cdot (0.92414)(0.07586) \cdot 1 (0.48535) + 1$$

$$\boxed{W_{h_1 x_2} = 1.0034026}$$

$$W_{h_2 x_2} = 0.1 \cdot 1 \cdot (0.26894)(0.73106) \cdot 0.5 (0.48535) - 3$$

$$\boxed{W_{h_2 x_2} = -2.995229}$$

$$W_{h_1 b} = 0.1 \cdot 1 \cdot (0.92414)(0.07586) \cdot 1 \cdot (0.48535) + 1.5$$

$$\boxed{W_{h_1 b} = 1.5034026}$$

$$W_{h_2 b} = 0.1 \cdot 1 \cdot (0.26894)(0.73106) \cdot 0.5 \cdot (0.48535) + 2$$

$$\boxed{W_{h_2 b} = 2.0047713}$$